Structural Properties of Markov Decision Processes

(CSC2547: Introduction to Reinforcement Learning)

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Goal

We study some important properties of value functions and MDPs.

- Bellman equation
- Bellman operator
  - Monotonicity
  - Contraction
- Focus on discounted tasks
- We show important consequences such as
  - The uniqueness of the solution to the Bellman equations
  - Error bounds on value error
  - Fixed point of T^* is the optimal value function

We refer to these frequently in studying and analyzing RL/Planning algorithms.

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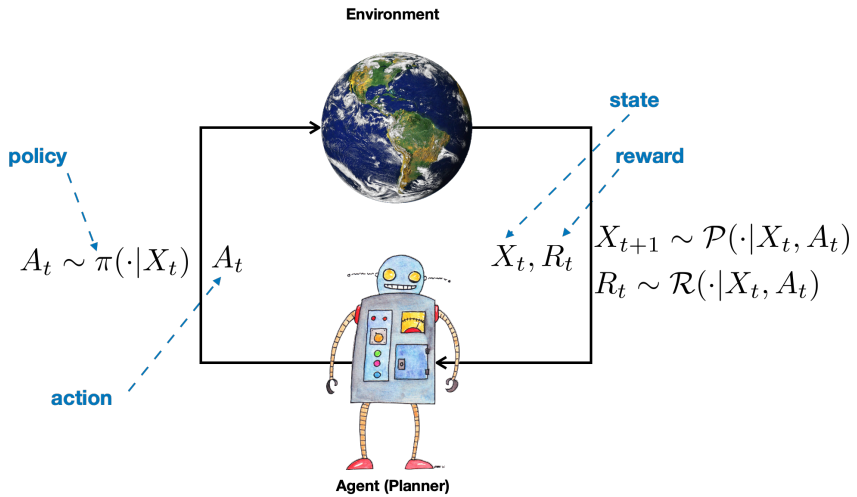
Return



Consider the sequence of rewards $(R_1, R_2, \dots)$ generated after the agent starts at state $X_1 = x$ and follows policy π. Given the discount factor $0 \leq \gamma < 1$, the return is

$$G_t^\pi \triangleq \sum_{k \geq t} \gamma^{k-t} R_k.$$

Recursive Property of Return

Comparing G_t^π and G_{t+1}^π, we observe that

$$G_t^\pi = R_t + \gamma G_{t+1}^\pi. \tag{1}$$

Interpretation: The return at the current time is equal to the immediate reward plus the *discounted* return at the next time step.

- Return is a random variable (r.v.).
- If we repeat the experiment from the same state x, the return would be different.
- Its distribution, however, is the same.
- Q: When would it be the same?

From Return to the Bellman Equation

We take (conditional) expectation of G_t^π (conditioned on state x), and expand the return as in (1):

$$\begin{aligned} V^\pi(x) &= \mathbb{E}\left[G_t^\pi \mid X_t = x\right] \\ &= \mathbb{E}\left[R_t + \gamma G_{t+1}^\pi \mid X_t = x\right] \\ &= \mathbb{E}\left[R(X_t, A_t) \mid X_t = x\right] + \gamma \mathbb{E}\left[G_{t+1}^\pi \mid X_t = x\right] \\ &= r^\pi(x) + \gamma \mathbb{E}\left[V^\pi(X_{t+1}) \mid X_t = x\right]. \end{aligned} \tag{2}$$

Neither side is random anymore!

Expanding $\mathbb{E}\left[V^\pi(X_{t+1}) \mid X_t = x\right]$

What does $\mathbb{E}\left[V^\pi(X_{t+1}) \mid X_t = x\right]$ mean?
It is the expected value of $V^\pi(X_{t+1})$ when

- the agent is at state x at time t
- chooses action $A \sim \pi(\cdot|x)$
- goes to a state $X_{t+1} \sim \mathcal{P}(\cdot|x, A)$

That is:

$$\mathbb{E}\left[V^\pi(X_{t+1}) \mid X_t = x\right] = \int \mathcal{P}(\mathrm{d}x'|x, a)\pi(\mathrm{d}a|x)V^\pi(x'). \tag{3}$$

For countable state-action spaces, we have

$$\mathbb{E}\left[V^\pi(X_{t+1}) \mid X_t = x\right] = \sum_{x',a} \mathcal{P}(x'|x, a)\pi(a|x)V^\pi(x').$$

Bellman Equation for a Policy π

By (2) and (3), we get that for any $x \in \mathcal{X}$, we have

$$V^\pi(x) = r^\pi(x) + \gamma \int \mathcal{P}(\mathrm{d}x'|x,a)\pi(\mathrm{d}a|x)V^\pi(x'). \qquad (4)$$

This is the Bellman equation for a policy π.
Interpretation: The value of following a policy π starting from the state x is the reward that the π-following agent receives at that state plus the discounted average (expected) value that the agent receives at the next-state.

Bellman Equation for a Policy π

Using the notation of $\mathcal{P}^\pi$:

$$V^\pi(x) = r^\pi(x) + \gamma \int \mathcal{P}^\pi(\mathrm{d}x'|x) V^\pi(x').$$

Or even more compactly,

$$V^\pi = r^\pi + \gamma \mathcal{P}^\pi V^\pi.$$

Remark

Recall that $(\mathcal{P}^\pi)(A|x) \triangleq \int_{\mathcal{X}} \mathcal{P}(\mathrm{d}y|x, \pi(x)) \mathbb{I}_{\{y \in A\}}$.

Bellman Equation for a Policy π (Q^π)

The Bellman equation for the action-value function Q^π:

$$Q^\pi(x,a) = r(x,a) + \gamma \int \mathcal{P}(\mathrm{d}x'|x,a) V^\pi(x')$$

$$= r(x,a) + \gamma \int \mathcal{P}(\mathrm{d}x'|x,a)\pi(\mathrm{d}a'|x') Q^\pi(x',a'). \quad (5)$$

More compactly:

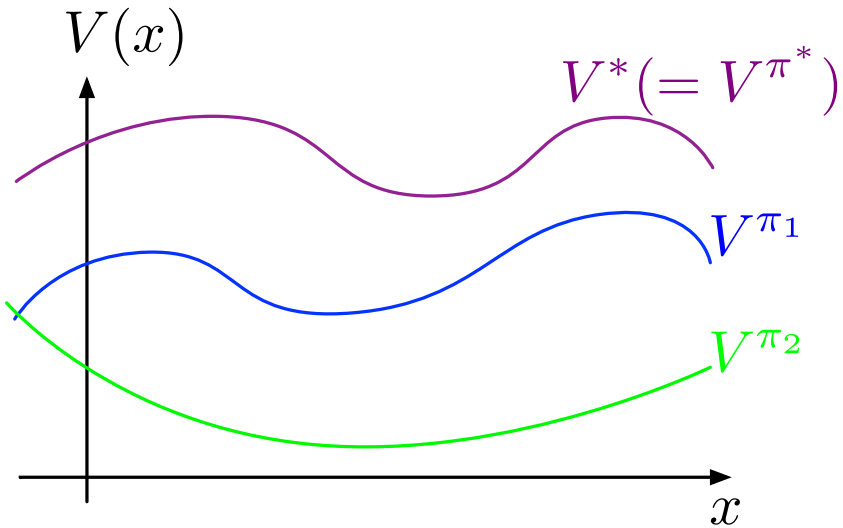
$$Q^\pi = r + \gamma \mathcal{P} V^\pi,$$

with the understanding that V^π and Q^π are related as $V^\pi(x) = \int \pi(\mathrm{d}a|x) Q^\pi(x,a)$.

Remark

The difference with the Bellman equation for V^π is that the choice of action at the first time step is pre-specified, instead of being selected by policy π.

Optimal Policy and Value Function



Recall that the optimal policy π^* is a policy that satisfies $\pi^* \geq \pi$ for any (stationary Markov) policy π. It satisfies

$$\pi^* \leftarrow \operatorname*{argmax}_{\pi \in \Pi} V^\pi.$$

Given an optimal policy, the optimal value function would be V^{π^*}.

Bellman Equations for Optimal Value Functions

Does the optimal value function V^{π^*} satisfy a recursive relation similar to the Bellman equation for a policy π?
Short answer: Yes!
But we have to be a bit careful. Why?! We have to go through a few steps of argument.

Bellman Equations for Optimal Value Functions

The argument goes through three claims:

1. There exists a unique value function V^* that satisfies the following equation: For any $x \in \mathcal{X}$, we have

$$V^*(x) = \max_{a \in \mathcal{A}} \left\{ r(x,a) + \gamma \int \mathcal{P}(\mathrm{d}x'|x,a) V^*(x') \right\}. \tag{6}$$

   This equation is called the Bellman optimality equation for the value function.
2. V^* is indeed the same as V^{π^*}, the optimal value function when π is restricted to be within the space of stationary policies.
3. For discounted continuing MDPs, we can always find a stationary policy that is optimal within the space of all stationary and non-stationary policies.

In summary: V^* exists and is equal to V^{π^*}.

Bellman Equations for Optimal Value Functions (Q^*)

Optimal action-value function:

$$Q^*(x,a) = r(x,a) + \gamma \int \mathcal{P}(\mathrm{d}x'|x,a) \max_{a' \in \mathcal{A}} Q^*(x',a'). \qquad (7)$$

Solutions of the Bellman Equations?

We have defined the Bellman equations for a fixed policy π and the Bellman optimality equation. Some reasonable questions:

- Is there only one solution V^π (or Q^π) satisfying (4) and (5)?
- Is there only one solution V^* (or Q^*) satisfying the Bellman optimality equations (6) and (7)?

We shall prove that their solutions are unique. We need some tools before doing so.

Optimal Policy from the Optimal Value Function

- If we know V^* or Q^*, we can find the optimal policy π^*.
- It is a deterministic policy.
- For any $x \in \mathcal{X}$, the optimal policy is

$$\begin{aligned}\pi^*(x) &= \operatorname*{argmax}_{a \in \mathcal{A}} Q^*(x,a) \\ &= \operatorname*{argmax}_{a \in \mathcal{A}} \left\{ r(x,a) + \gamma \int \mathcal{P}(\mathrm{d}x'|x,a) V^*(x') \right\}.\end{aligned}$$

Optimal Policy from the Optimal Value Function

$$\pi^*(x) = \operatorname*{argmax}_{a \in \mathcal{A}} \left\{ r(x,a) + \gamma \int \mathcal{P}(\mathrm{d}x'|x,a) V^*(x') \right\}.$$

Interpretation: Suppose that the agent is at state x. To act optimally,

- It needs to act optimality both at the current time step (Now) and in the Future time steps.
- Suppose that we know that the agent is going to act optimally in the Future. This means that when it get to the next state $X' \sim \mathcal{P}(\cdot|x,a)$,
  - it follows the optimal policy π^*.
  - The value of following the optimal policy is going to be $V^*(X')$.
- (continued ...)

Optimal Policy from the Optimal Value Function

$$\pi^*(x) = \operatorname*{argmax}_{a \in \mathcal{A}} \left\{ r(x,a) + \gamma \int \mathcal{P}(\mathrm{d}x'|x,a) V^*(x') \right\}.$$

Interpretation: Suppose that the agent is state x. To act optimally,

- ...
- Since we do not know where the agent will be at the next time step, the expected performance of acting optimally in the Future is $\int \mathcal{P}(\mathrm{d}x'|x,a) V^*(x')$.
- As we are dealing with discounted tasks, the performance of the agent at the current state x is going to be $r(x,a) + \gamma \int \mathcal{P}(\mathrm{d}x'|x,a) V^*(x')$.
- To act optimally Now, the agent should choose an action that maximizes this value.

Greedy Policy

The mapping that selects an action by choosing the maximizer of the (action-) value function is called the greedy policy.

- For $Q \in \mathcal{B}(\mathcal{X} \times \mathcal{A})$, the greedy policy $\pi_g : \mathcal{X} \times \mathcal{B}(\mathcal{X} \times \mathcal{A}) \to \mathcal{A}$ is

$$\pi_g(x; Q) = \operatorname*{argmax}_{a \in \mathcal{A}} Q(x, a).$$

- For $V \in \mathcal{B}(\mathcal{X})$, the greedy policy is

$$\pi_g(x; V) = \operatorname*{argmax}_{a \in \mathcal{A}} \left\{ r(x, a) + \gamma \int \mathcal{P}(\mathrm{d}x' | x, a) V(x') \right\}.$$

- We use $\pi_g(V)$ and $\pi_g(Q)$ to denote functions from $\mathcal{X}$ to $\mathcal{A}$.
- $\pi_g(V^*) = \pi_g(Q^*) = \pi^*$.

Greedy Policy

Intuition behind the greedy policy:

- Action selection based on the local information.
- Does not look at all future possibilities
- Only one step ahead (for V) or even no-step ahead (for Q) in order to pick the action. This is myopic.
- Given V^* or Q^*, however, the selected action is going to be the optimal one.
- This is because the optimal value functions encodes the information about the future, so we do not need to explicitly consider all possible futures.

Bellman Operators

- The Bellman equations can be seen as the fixed point equation of certain operators known as the Bellman operators.
  - What this means become clear soon.
- Recall that an operator (or mapping) $L : \mathcal{Z} \to \mathcal{Z}$ takes a member of space $\mathcal{Z}$ and returns another member of $\mathcal{Z}$.
  - If $\mathcal{Z} = \mathbb{R}$ and $L : z \mapsto z^2$. So $L(5) = 25$ (this is the usual function).
  - If $\mathcal{Z}$ is the space of smooth functions defined on domain $\mathbb{R}$, $L : z \mapsto \frac{d}{dx} z$, is the differentiation operator. So $L(\sin(x)) = \cos(x)$.

Bellman Operators

Definition (Bellman Operators for policy π)

Given a policy $\pi : \mathcal{X} \to \mathcal{M}(\mathcal{A})$, the Bellman operators $T^\pi : \mathcal{B}(\mathcal{X}) \to \mathcal{B}(\mathcal{X})$ and $T^\pi : \mathcal{B}(\mathcal{X} \times \mathcal{A}) \to \mathcal{B}(\mathcal{X} \times \mathcal{A})$ are defined as the mapping

$$(T^\pi V)(x) \triangleq r^\pi(x) + \gamma \int \mathcal{P}(\mathrm{d}x'|x,a)\pi(\mathrm{d}a|x)V(x'),$$

$$(T^\pi Q)(x,a) \triangleq r(x,a) + \gamma \int \mathcal{P}(\mathrm{d}x'|x,a)\pi(\mathrm{d}a'|x')Q(x',a'),$$

defined for all $x \in \mathcal{X}$ (for V) or all $(x,a) \in \mathcal{X} \times \mathcal{A}$ (for Q).

Bellman Operators

If π is deterministic:

$$(T^\pi V)(x) \triangleq r^\pi(x) + \gamma \int \mathcal{P}(\mathrm{d}x'|x, \pi(x))V(x'),$$

$$(T^\pi Q)(x,a) \triangleq r(x,a) + \gamma \int \mathcal{P}(\mathrm{d}x'|x, a)Q(x', \pi(x')).$$

Bellman Operators and Bellman Equation

Recall that

$$V^\pi(x) = r^\pi(x) + \gamma \int \mathcal{P}(\mathrm{d}x'|x,a)\pi(\mathrm{d}a|x)V^\pi(x'),$$

$$Q^\pi(x,a) = r(x,a) + \gamma \int \mathcal{P}(\mathrm{d}x'|x,a)\pi(\mathrm{d}a'|x')Q^\pi(x',a').$$

Using the Bellman operator T^π, we can write them compactly as

$$V^\pi = T^\pi V^\pi,$$
$$Q^\pi = T^\pi Q^\pi.$$

This is a compact form of Bellman equations.

Bellman Optimality Operators

Definition (Bellman Optimality Operators)

The Bellman operators $T^* : \mathcal{B}(\mathcal{X}) \to \mathcal{B}(\mathcal{X})$ and $T^* : \mathcal{B}(\mathcal{X} \times \mathcal{A}) \to \mathcal{B}(\mathcal{X} \times \mathcal{A})$ are defined as the mapping

$$(T^*V)(x) \triangleq \max_{a \in \mathcal{A}} \left\{ r(x,a) + \gamma \int \mathcal{P}(\mathrm{d}x'|x,a) V(x') \right\},$$

$$(T^*Q)(x,a) \triangleq r(x,a) + \gamma \int \mathcal{P}(\mathrm{d}x'|x,a) \max_{a' \in \mathcal{A}} Q(x',a'),$$

defined for all $x \in \mathcal{X}$ (for V) or all $(x,a) \in \mathcal{X} \times \mathcal{A}$ (for Q).

Bellman Optimality Operators and Bellman Optimality Equation

Comparing with

$$V^*(x) = \max_{a \in \mathcal{A}} \left\{ r(x,a) + \gamma \int \mathcal{P}(\mathrm{d}x'|x,a) V^*(x') \right\},$$

$$Q^*(x,a) = r(x,a) + \gamma \int \mathcal{P}(\mathrm{d}x'|x,a) \max_{a' \in \mathcal{A}} Q^*(x',a'),$$

we see that

$$V^* = T^* V^*,$$

$$Q^* = T^* Q^*.$$

Properties of the Bellman Operators

The Bellman operators have some important properties. The properties that matters for us the most are

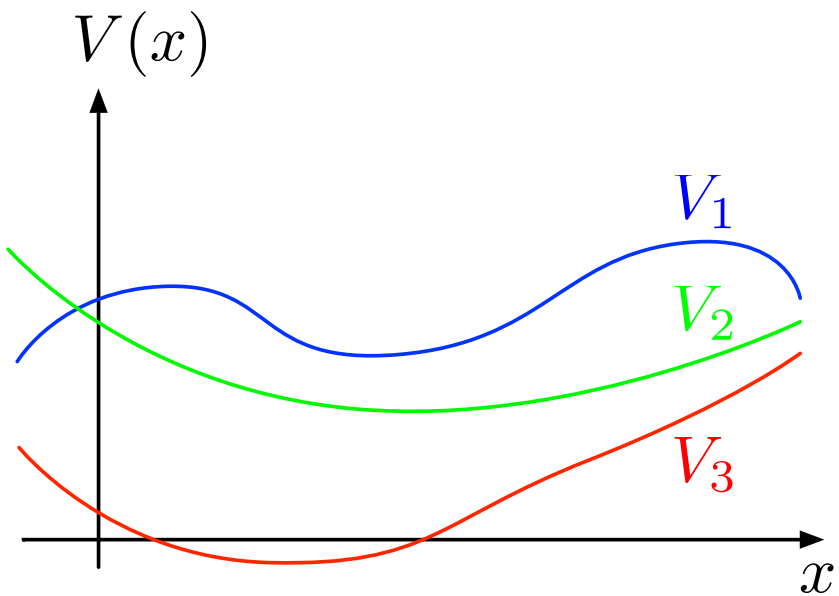
- Monotonicity
- Contraction

They are used in

- basic proofs such as the existence and uniqueness of the solution to the Bellman equations.
- (directly or indirectly) design of many RL/Planning algorithms.

Monotonicity

For two functions $V_1, V_2 \in \mathcal{B}(\mathcal{X})$, we use $V_1 \leq V_2$ if and only if $V_1(x) \leq V_2(x)$ for all $x \in \mathcal{X}$.



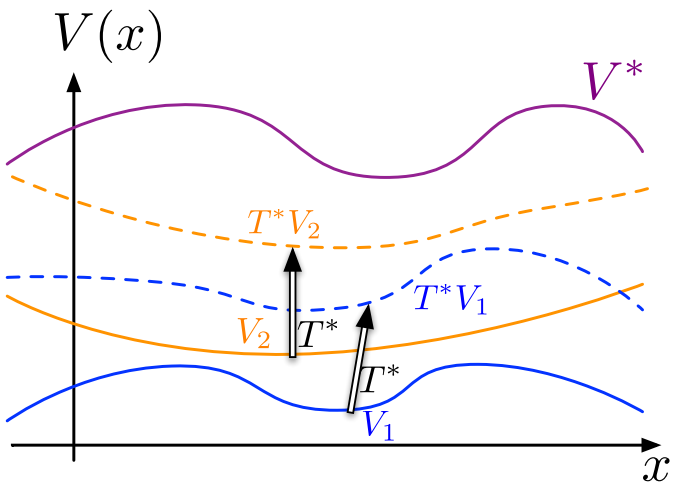
- $V_3 \leq V_1$ and $V_3 \leq V_2$,
- Neither $V_2 \leq V_1$, nor $V_1 \leq V_2$.

Monotonicity

Lemma (Monotonicity)

Fix a policy π. If $V_1, V_2 \in \mathcal{B}(\mathcal{X})$, and $V_1 \leq V_2$, then we have

$$T^\pi V_1 \leq T^\pi V_2,$$
$$T^* V_1 \leq T^* V_2.$$



Monotonicity (Proof)

We only prove the first claim. Let us expand $T^\pi V_1$. As $V_1(x') \le V_2(x')$ for any $x' \in \mathcal{X}$, we get that for any $x \in \mathcal{X}$,

$$(T^\pi V_1)(x) = r^\pi(x) + \gamma \int \mathcal{P}^\pi(\mathrm{d}x'|x) \underbrace{V_1(x')}_{\le V_2(x')}$$

$$\le r^\pi(x) + \gamma \int \mathcal{P}^\pi(\mathrm{d}x'|x) V_2(x') = (T^\pi V_2)(x).$$

Therefore, $T^\pi V_1 \le T^\pi V_2$.

Contraction Mapping and Banach Fixed Point Theorem

Let us review some mathematical background before proving that the Bellman operators are contraction. We quote several results from Hunter and Nachtergaele [2001].

Metric

Definition (Metric)

A metric or a distance function on $\mathcal{Z}$ is a function $d : \mathcal{Z} \times \mathcal{Z} \to \mathbb{R}$ with the following properties:

- $d(x, y) \geq 0$ for all $x, y \in \mathcal{Z}$; and $d(x, y) = 0$ if and only if $x = y$.
- $d(x, y) = d(y, x)$ for all $x, y \in \mathcal{Z}$ (symmetry).
- $d(x, y) \leq d(x, z) + d(z, y)$ for all $x, y, z \in \mathcal{Z}$ (triangle inequality).

A metric space $(\mathcal{Z}, d)$ is a set $\mathcal{Z}$ equipped with a metric d.

Example

Let $\mathcal{Z} = \mathbb{R}$ and $d(x, y) = |x - y|$. These together define a metric space $(\mathbb{R}, d)$.

Norm

Definition (Norm)

A norm on a linear space $\mathcal{Z}$ is a function $\|\cdot\| : \mathcal{Z} \to \mathbb{R}$ with the following properties:

- (non-negative) For all $x \in \mathcal{Z}$, $\|x\| \geq 0$.
- (homogenous) For all $x \in \mathcal{Z}$ and $\lambda \in \mathbb{R}$, $\|\lambda x\| = |\lambda| \, \|x\|$.
- (triangle inequality) For all $x, y \in \mathcal{Z}$, $\|x + y\| \leq \|x\| + \|y\|$.
- (strictly positive) If for a $x \in \mathcal{Z}$, we have that $\|x\| = 0$, it implies that $x = 0$.

Remark

We can use a norm to define a distance between two points in a linear space $\mathcal{Z}$ by defining $d(x, y) = \|x - y\|$. This gives us a metric space $(\mathcal{Z}, d)$.

Norm

Example

Let $\mathcal{Z} = \mathbb{R}^d$ $(d \geq 1)$. The following norms are often used:

$$\|x\|_p = \sqrt[p]{\sum_{i=1}^{d} |x_i|^p}, \qquad 1 \leq p < \infty,$$

$$\|x\|_\infty = \max_{i=1,\ldots,d} |x_i|.$$

Norm

Example

Consider the space of continuous functions with domain $[0, 1]$. It is denoted by $\mathcal{C}([0, 1])$. This plays the rule of $\mathcal{Z}$. We define the following norm for a function $f \in \mathcal{C}([0, 1])$:

$$\|f\|_\infty = \sup_{x \in [0,1]} |f(x)|.$$

This is called the supremum or uniform norm. Given this norm, $(\mathcal{C}([0, 1]), \|\cdot\|_\infty)$ would be a normed linear space.

Norm

For $V \in \mathcal{B}(\mathcal{X})$ and $Q \in \mathcal{B}(\mathcal{X} \times \mathcal{A})$, their supremum norms are

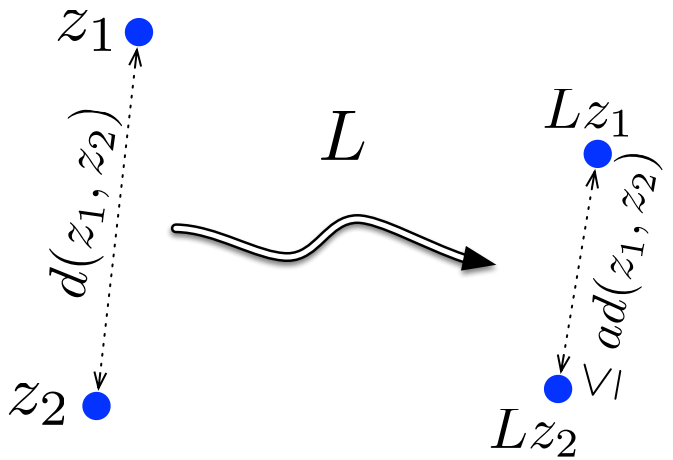
$$\|V\|_\infty = \sup_{x \in \mathcal{X}} |V(x)|,$$
$$\|Q\|_\infty = \sup_{(x,a) \in \mathcal{X} \times \mathcal{A}} |Q(x,a)|,$$

Contraction Mapping

Definition (Contraction Mapping)

Let $(\mathcal{Z}, d)$ be a metric space. A mapping $L : \mathcal{Z} \to \mathcal{Z}$ is a contraction mapping (or contraction) if there exists a constant $0 \leq a < 1$ such that for all $z_1, z_2 \in \mathcal{Z}$, we have

$$d(L(z_1), L(z_2)) \leq a d(z_1, z_2).$$



Contraction Mapping

Example

Let $\mathcal{Z} = \mathbb{R}$ and $d(z_1, z_2) = |z_1 - z_2|$. Consider the mapping $L : z \mapsto az$ for $a \in \mathbb{R}$.
For any $z_1, z_2 \in \mathbb{R}$, we have

$$\begin{aligned} d(L(z_1), L(z_2)) &= |L(z_1) - L(z_2)| = |az_1 - az_2| \\ &= |a||z_1 - z_2| = |a|d(z_1, z_2). \end{aligned}$$

So if $|a| < 1$, this is a contraction mapping.

Why Do We Care About Contraction Mapping?

Two main reasons:

- It describes the stability behaviour of a dynamical system.
- Sometimes can be used to solve equations.

Why Do We Care About Contraction Mapping?

As an example of its relation to stability, let $z_0 \in \mathcal{Z}$ and consider a mapping $L : z \mapsto az$ for some $a \in \mathbb{R}$. Define the dynamical system

$$z_{k+1} = Lz_k, \qquad k = 0, 1, \dots .$$

The dynamical system described by this mapping generates

$$\begin{aligned} z_0 & \\ z_1 &= az_0 \\ z_2 &= az_1 = a^2 z_0 \\ &\vdots \\ z_k &= az_{k-1} = a^k z_0. \end{aligned}$$

Why Do We Care About Contraction Mapping?

$$z_k = a^k z_0.$$

- If $|a| < 1$, z_k converges to zero, no matter what z_0 is.
- If $a = 1$, we have $z_k = z_0$. So depending on z_0, it converges to different points.
- For $a = -1$, the sequence would oscillate between $+z_0$ and $-z_0$.
- If $|a| > 1$, the sequence diverges (unless $z_0 = 0$).

Remark

The case of converge is the same as the case of L being a contraction map.

Why Do We Care About Contraction Mapping?

Definition (Fixed Point)

If $L : \mathcal{Z} \to \mathcal{Z}$, then a point $z \in \mathcal{Z}$ such that

$$Lz = z$$

is called a fixed point of L.

Given an equation $f(z) = 0$, we can convert it to a fixed point equation $Lz = z$ by defining $L : z \mapsto f(z) + z$. Then, if $Lz^* = z^*$ for a z^*, we get that $f(z^*) = 0$, i.e., the fixed point of L is the same as the solution of $f(z) = 0$.

Banach Fixed Point Theorem

Theorem (Banach Fixed Point Theorem)

If $L : \mathcal{Z} \to \mathcal{Z}$ is a contraction mapping on a complete metric space (Z, d), then there exists a unique $z^* \in \mathcal{Z}$ such that $Lz^* = z^*$. Furthermore, the point z^* can be found by choosing an arbitrary $z_0 \in \mathcal{Z}$ and defining $z_{k+1} = Lz_k$. We have $z_k \to z^*$.

Simple Exercise

Exercise

Suppose that we want to solve $cz + b = 0$ *for* $z \in \mathbb{R}$ *and constants* $c, b \in \mathbb{R}$.

- *Choose a mapping* $L : \mathbb{R} \to \mathbb{R}$ *such that its fixed point is the same as the solution of this equation.*
- *For what range of* c *is this mapping a contraction?*
- *Let* $c = -0.5$ *and* $b = 1$. *If we start from* $z_0 = 0$, *what is the sequence of* z_0, z_1, z_2 *that we obtain by computing* $z_{k+1} = Lz_k$*?*

Bellman Operator is a Contraction

Lemma (Contraction)

For any π, the Bellman operator T^π is a γ-contraction mapping.
The Bellman optimality operator T^* is a γ-contraction mapping.

Bellman Operator is a Contraction (Proof)

We only show it for the Bellman operator $T^\pi : \mathcal{B}(\mathcal{X} \times \mathcal{A}) \to \mathcal{B}(\mathcal{X} \times \mathcal{A})$.

Consider two action-value functions $Q_1, Q_2 \in \mathcal{B}(\mathcal{X} \times \mathcal{A})$. Consider the metric $d(Q_1, Q_2) = \|Q_1 - Q_2\|_\infty$. We show the contraction w.r.t. this metric.

For any $(x, a) \in \mathcal{X} \times \mathcal{A}$, we have

$$
\begin{aligned}
&|(T^\pi Q_1)(x,a) - (T^\pi Q_2)(x,a)| = \\
&\left| \left[r(x,a) + \gamma \int \mathcal{P}(\mathrm{d}x'|x,a)\pi(\mathrm{d}a'|x')Q_1(x',a') \right] - \right. \\
&\left. \left[r(x,a) + \gamma \int \mathcal{P}(\mathrm{d}x'|x,a)\pi(\mathrm{d}a'|x')Q_2(x',a') \right] \right| \\
&= \gamma \left| \int \mathcal{P}(\mathrm{d}x'|x,a)\pi(\mathrm{d}a'|x') \left(Q_1(x',a') - Q_2(x',a') \right) \right| .
\end{aligned}
$$

Bellman Operator is a Contraction (Proof)

Let us upper bound the RHS.
We have an integral of the form $\left|\int P(\mathrm{d}x) f(x)\right|$ (or a summation $|\sum_x P(x) f(x)|$ for a countable state space). This can be upper bounded as

$$\begin{aligned}\left|\int P(\mathrm{d}x) f(x)\right| &\leq \int |P(\mathrm{d}x) f(x)| = \int |P(\mathrm{d}x)|.|f(x)| \\ &\leq \int P(\mathrm{d}x). \sup_{x\in\mathcal{X}} |f(x)| \\ &= \sup_{x\in\mathcal{X}} |f(x)| \int P(\mathrm{d}x) = \|f\|_\infty ,\end{aligned}$$

where we used $\int P(\mathrm{d}x) = 1$.

Bellman Operator is a Contraction (Proof)

In our case, we get that

$$
\begin{aligned}
&|(T^\pi Q_1)(x,a) - (T^\pi Q_2)(x,a)| = \\
&\gamma \left| \int \mathcal{P}(\mathrm{d}x'|x,a)\pi(\mathrm{d}a'|x') \left(Q_1(x',a') - Q_2(x',a')\right) \right| \\
&\leq \gamma \int \mathcal{P}(\mathrm{d}x'|x,a)\pi(\mathrm{d}a'|x') \left|Q_1(x',a') - Q_2(x',a')\right| \\
&\leq \gamma \left\|Q_1 - Q_2\right\|_\infty \int \mathcal{P}(\mathrm{d}x'|x,a)\pi(\mathrm{d}a'|x') \\
&= \gamma \left\|Q_1 - Q_2\right\|_\infty .
\end{aligned}
$$

This inequality holds for any $(x,a) \in \mathcal{X} \times \mathcal{A}$, so it holds for its supremum over $\mathcal{X} \times \mathcal{A}$ too, i.e.,

$$\left\|(T^\pi Q_1) - (T^\pi Q_2)\right\|_\infty \leq \gamma \left\|Q_1 - Q_2\right\|_\infty .$$

This shows that T^π is a γ-contraction.

Consequences of Monotonicity and Contraction

Bellman operators are

- Monotonic
- γ-contraction

Several consequences:

- Bellman equations have unique fixed points.
- Error bounds on the difference between V and V^* when $V \approx T^*V$.
- V^* is the optimal value function V^{π^*}.

Uniqueness of Fixed Points

Proposition (Uniqueness of Fixed Points)

The operators T^π and T^* have unique fixed points, denoted by V^π and V^*, i.e.,

$$V^\pi = T^\pi V^\pi,$$
$$V^* = T^* V^*.$$

They can be computed from any $V_0 \in \mathcal{B}(\mathcal{X})$ by iteratively computing $V_{k+1} \leftarrow T^* V_k$ (and similar for V^π using T^π instead) for $k = 0, 1, \ldots$. We have that $V_k \to V^*$ (and similarly, $V_k \to V^\pi$).

The same result is true for Q^π and Q^*.

Uniqueness of Fixed Points (Proof)

- Consider the space of bounded functions $\mathcal{B}(\mathcal{X})$ with the metric d based on the uniform norm, i.e., $d(V_1, V_2) = \|V_1 - V_2\|_\infty$. The space $(\mathcal{B}(\mathcal{X}), d)$ is a complete metric space.
- For any π, the operator T^π is a γ-contraction. Likewise, T^* has the same property too (Lemma 13).
- By the Banach fixed point theorem (Theorem 12), they have a unique fixed point. Moreover, any sequence (V_k) with $V_0 \in \mathcal{B}(\mathcal{X})$ and $V_{k+1} \leftarrow T^\pi V_k$ $(k = 0, 1, \dots)$ is convergent, which means that $\lim_{k\to\infty} \|V_k - V^\pi\|_\infty = 0$.

Value of the Greedy Policy of V^* is V^*

Proposition

We have $T^\pi V^* = T^* V^*$ if and only if $V^\pi = V^*$.

Remark

The statement and the proof is from Proposition 2.1.1(c) of Bertsekas 2018.

Value of the Greedy Policy of V^* is V^* (Proof)

Proof of $T^\pi V^* = T^* V^* \Longrightarrow V^\pi = V^*$:
Assume that $T^\pi V^* = T^* V^*$.
As V^* is the solution of the Bellman optimality equation, we have $T^* V^* = V^*$. Therefore,

$$T^\pi V^* = T^* V^* = V^*.$$

This shows that V^* is a fixed point of T^π.
The fixed point of T^π, however, is unique (Proposition 14) and is equal to V^π.
So V^π and V^* should be the same, i.e., $V^\pi = V^*$.

Value of the Greedy Policy of V^* is V^* (Proof)

Proof of $V^\pi = V^* \Longrightarrow T^\pi V^* = T^* V^*$:
We apply T^π to both sides of $V^* = V^\pi$ to get

$$T^\pi V^* = T^\pi V^\pi.$$

As V^π is the solution of the Bellman equation for policy π, we have $T^\pi V^\pi = V^\pi$. Therefore,

$$T^\pi V^* = T^\pi V^\pi = V^\pi.$$

By assumption, $V^\pi = V^*$. So we have $T^\pi V^* = V^\pi = V^*$.
On the other hand, we have $V^* = T^* V^*$, so

$$T^\pi V^* = V^* = T^* V^*,$$

which is the desired result.

Value of the Greedy Policy of V^* is V^*

Discussion:

- If $T^\pi V^* = T^* V^*$ for some policy π, the value function V^π of that policy is the same as the fixed point of T^*, which is V^*.
- We have not yet shown that the fixed point of T^* is an optimal value function, in the sense that it is $\pi^* \leftarrow \operatorname{argmax}_{\pi \in \Pi} V^\pi(x)$ (for all $x \in \mathcal{X}$) over the space of all stationary policies Π (or even more generally, over the set of all non-stationary policies)
- But it is indeed true!

Value of the Greedy Policy of V^* is V^*

To see the connection to the greedy policy:

- Given V^*, the greedy policy selects $\pi_g(x; V^*) = \text{argmax}_{a \in \mathcal{A}} \left\{ r(x,a) + \gamma \int \mathcal{P}(\mathrm{d}x'|x,a) V^*(x') \right\}$.
- So $T^{\pi_g(V^*)} V^* = \max_{a \in \mathcal{A}} \left\{ r(x,a) + \gamma \int \mathcal{P}(\mathrm{d}x'|x,a) V^*(x') \right\}$
- Compare with $T^* V^*$, i.e., $(T^* V^*)(x) = \max_{a \in \mathcal{A}} \left\{ r(x,a) + \gamma \int \mathcal{P}(\mathrm{d}x'|x,a) V^*(x') \right\}$.
- So $T^{\pi_g(V^*)} V^* = T^* V^*$.
- This proposition states that the value of following $\pi_g(V^*)$, that is $V^{\pi_g(V^*)}$, is the same as V^*.
- The practical consequence is that if we find V^* and its greedy policy $\pi_g(V^*)$, the value of following the greedy is V^*.
- Practical Consequence: To find an optimal policy, we can find V^* first and then follow its greedy policy $\pi_g(V^*)$.

What if $V \approx T^*V$?

- If we find a V such that $V = T^*V$, we know that $V = V^*$ (similar for T^π and Q).
- What if $V \approx T^*V$? What can be said about the closeness of V to V^*?
- Practically important, because we often can only solve the Bellman equations approximately, because of various sources of errors
  - Computational
  - Approximation
  - Statistical

An Error Bound based on the Bellman Error

Proposition

For any $V \in \mathcal{B}(\mathcal{X})$ or $Q \in \mathcal{B}(\mathcal{X} \times \mathcal{A})$, we have

$$\|V - V^*\|_\infty \leq \frac{\|V - T^*V\|_\infty}{1-\gamma}, \qquad \|Q - Q^*\|_\infty \leq \frac{\|Q - T^*Q\|_\infty}{1-\gamma}.$$

The quantity $\mathrm{BR}(V) \triangleq T^\pi V - V$ and $\mathrm{BR}^*(V) \triangleq T^*V - V$ are called Bellman Residuals.
Their norms are called Bellman Errors.

An Error Bound based on the Bellman Error (Proof)

We want to upper bound $\|V - V^*\|_\infty$.
We start from $V - V^*$, and add and subtract T^*V to $V - V^*$.
We then take the supremum norm, and use the triangle inequality to get

$$\begin{aligned} V - V^* &= V - T^*V + T^*V - V^* \\ \Rightarrow \|V - V^*\|_\infty &= \|V - T^*V + T^*V - V^*\|_\infty \\ &\leq \|V - T^*V\|_\infty + \|T^*V - V^*\|_\infty . \end{aligned}$$

An Error Bound based on the Bellman Error (Proof)

Let us focus on the term $\|T^*V - V^*\|_\infty$. Two observations:

- $V^* = T^*V^*$.
- The Bellman optimality operator is a γ-contraction w.r.t. the supremum norm.

Thus,

$$\|T^*V - V^*\|_\infty = \|T^*V - T^*V^*\|_\infty \leq \gamma \|V - V^*\|_\infty .$$

Therefore,

$$\|V - V^*\|_\infty \leq \|V - T^*V\|_\infty + \gamma \|V - V^*\|_\infty .$$

Re-arranging this, we get

$$(1 - \gamma) \|V - V^*\|_\infty \leq \|V - T^*V\|_\infty .$$

An Error Bound based on the Bellman Error (for policy π)

Proposition

For any $V \in \mathcal{B}(\mathcal{X})$ or $Q \in \mathcal{B}(\mathcal{X} \times \mathcal{A})$, and any $\pi \in \Pi$, we have

$$\|V - V^\pi\|_\infty \leq \frac{\|V - T^\pi V\|_\infty}{1-\gamma}, \qquad \|Q - Q^\pi\|_\infty \leq \frac{\|Q - T^\pi Q\|_\infty}{1-\gamma}.$$

V^* is the same as V^{π^*}

The fixed point of T^* is indeed the optimal value function within the space of stationary policies Π.
We use the monotonicity of T^*, in addition to contraction, to prove it.

V^* is the same as V^{π^*}

Proposition

Let V^* be the fixed point of T^*, i.e., $V^* = T^* V^*$. We have

$$V^*(x) = \sup_{\pi \in \Pi} V^\pi(x), \qquad \forall x \in \mathcal{X}.$$

Remark

The statement and the proof is from Proposition 2.1.1 of Bertsekas 2018.

V^* is the same as V^{π^*} (Proof)

Overview:

- We show that $V^*(x) \leq \sup_{\pi \in \Pi} V^\pi(x)$.
- We show that $\sup_{\pi \in \Pi} V^\pi(x) \leq V^*(x)$.
- Combined, they show that $V^*(x) = \sup_{\pi \in \Pi} V^\pi(x)$.

V^* is the same as V^{π^*} (Proof)

Proof of $V^*(x) \leq \sup_{\pi \in \Pi} V^\pi(x)$:
From the error bound result (Proposition 17) with the choice of $V = V^*$, we get that for any $\pi \in \Pi$,

$$\|V^* - V^\pi\|_\infty \leq \frac{\|V^* - T^\pi V^*\|_\infty}{1-\gamma}. \tag{8}$$

Let $\varepsilon > 0$. Choose a policy π_ε such that

$$\|V^* - T^{\pi_\varepsilon} V^*\|_\infty \leq (1-\gamma)\varepsilon.$$

This is possible because we have

$$(T^* V^*)(x) = \sup_{a \in \mathcal{A}} \left\{ r(x,a) + \gamma \int \mathcal{P}(\mathrm{d}x'|x,a) V^*(x') \right\},$$

so it is sufficient to pick a π_ε that solves the optimization problem up to $(1-\gamma)\varepsilon$-accuracy of the supremum at each state x (if we find the maximizer, then $\varepsilon = 0$).

V^* is the same as V^{π^*} (Proof)

Proof of $V^*(x) \leq \sup_{\pi \in \Pi} V^\pi(x)$ (Continued):
For policy π_ε, (8) shows that

$$\|V^* - V^{\pi_\varepsilon}\|_\infty \leq \varepsilon.$$

This means that

$$V^*(x) \leq V^{\pi_\varepsilon}(x) + \varepsilon, \qquad \forall x \in \mathcal{X}.$$

Notice that $V^{\pi_\varepsilon}(x) \leq \sup_{\pi \in \Pi} V^\pi(x)$ (as $\pi_\varepsilon \in \Pi$). We take $\varepsilon \to 0$ to get that for all $x \in \mathcal{X}$,

$$V^*(x) \leq \lim_{\varepsilon \to 0} \{V^{\pi_\varepsilon}(x) + \varepsilon\} \leq \lim_{\varepsilon \to 0} \left\{ \sup_{\pi \in \Pi} V^\pi(x) + \varepsilon \right\} = \sup_{\pi \in \Pi} V^\pi(x). \tag{9}$$

This shows that V^*, the fixed point of T^*, is smaller or equal to the optimal value function within the space of stationary policies.

V^* is the same as V^{π^*} (Proof)

Proof of $\sup_{\pi \in \Pi} V^\pi(x) \leq V^*(x)$**:**
Consider any $\pi \in \Pi$. By the definition of T^π and T^*, for any $V \in \mathcal{B}(\mathcal{X})$, we have that for any $x \in \mathcal{X}$,

$$\begin{aligned}(T^\pi V)(x) &= \int \pi(\mathrm{d}a|x) \left[r(x,a) + \gamma \int \mathcal{P}(\mathrm{d}x'|x,a) V(x') \right] \\ &\leq \sup_{a \in \mathcal{A}} \left\{ r(x,a) + \gamma \int \mathcal{P}(\mathrm{d}x'|x,a) V(x') \right\} \\ &= (T^* V)(x).\end{aligned}$$

In particular, with the choice of $V = V^*$, we have

$$T^\pi V^* \leq T^* V^*.$$

V^* is the same as V^{π^*} (Proof)

Proof of $\sup_{\pi \in \Pi} V^{\pi}(x) \leq V^*(x)$ (Continued):

$$T^{\pi} V^* \leq T^* V^*.$$

As $T^* V^* = V^*$, we have

$$T^{\pi} V^* \leq V^*. \tag{10}$$

We use the monotonicity of T^{π} (Lemma 3) to conclude that

$$T^{\pi}(T^{\pi} V^*) \leq T^{\pi} V^*,$$

which by (10) shows that

$$(T^{\pi})^2 V^* \leq V^*.$$

We repeat this argument for k times to get that

$$(T^{\pi})^k V^* \leq V^*.$$

V^* is the same as V^{π^*} (Proof)

Proof of $\sup_{\pi\in\Pi} V^\pi(x) \leq V^*(x)$ (Continued):

$$(T^\pi)^k V^* \leq V^*.$$

As $k \to \infty$, Proposition 14 shows that $(T^\pi)^k V^*$ converges to V^π (the choice of V^* is irrelevant). Therefore,

$$V^\pi = \lim_{k\to\infty} (T^\pi)^k V^* \leq V^*.$$

As this holds for any $\pi \in \Pi$, we take the supremum over $\pi \in \Pi$ to get

$$\sup_{\pi\in\Pi} V^\pi \leq V^*. \tag{11}$$

Inequalities (9) and (11) together show the desired result.

Summary

- Bellman equations describe an important recursive properties of value functions.
- Bellman operators T^π and T^*.
- Greedy policy and the optimal policy.
- Monotonicity and contraction properties of the Bellman operators.
- Bellman equations have uniqued solutions.
- Bellman error $\|V - T^*V\|_\infty$ provides an upper bound on value error $\|V - V^*\|_\infty$.
- The solution of the Bellman optimality equation is the optimal value function.

References

Dimitri P. Bertsekas. *Abstract dynamic programming*. Athena Scientific Belmont, 2nd edition, 2018.

John K. Hunter and Bruno Nachtergaele. *Applied analysis*. World Scientific Publishing Company, 2001.