# Probability Theory Review

Introduction to Machine Learning (CSC 311)
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University of Toronto

# Motivation

Uncertainty arises through:

- Noisy measurements
- Variability between samples
- Finite size of data sets

Probability provides a consistent framework for the quantification and manipulation of uncertainty.

# Sample Space

- Sample space $\Omega$ is the set of all possible outcomes of an experiment.
- Observations $\omega \in \Omega$ are points in the space also called sample outcomes, realizations, or elements.
- Events $E \subset \Omega$ are subsets of the sample space.

In this experiment we flip a coin twice:

- Sample space All outcomes $\Omega = \{HH, HT, TH, TT\}$
- Observation $\omega = HT$ valid sample since $\omega \in \Omega$
- Event Both flips same $E = \{HH, TT\}$ valid event since $E \subset \Omega$

# Probability

The probability of an event E, $P(E)$, satisfies three axioms:

1: $P(E) \geq 0$ for every $E$

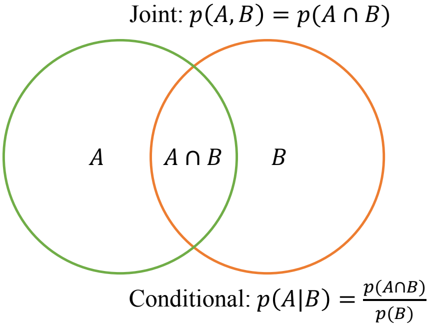
2: $P(\Omega) = 1$

3: If $E_1, E_2, \ldots$ are disjoint then

$$P(\bigcup_{i=1}^{\infty} E_i) = \sum_{i=1}^{\infty} P(E_i)$$

# Joint and Conditional Probabilities

Joint Probability of $A$ and $B$ is denoted $P(A, B)$.
Conditional Probability of $A$ given $B$ is denoted $P(A|B)$.




$$p(A, B) = p(A|B)p(B) = p(B|A)p(A)$$

# Conditional Example

Probability of passing the midterm is 60% and probability of passing both the final and the midterm is 45%.
What is the probability of passing the final given the student passed the midterm?

$$\begin{aligned} P(F|M) &= P(M,F)/P(M) \\ &= 0.45/0.60 \\ &= 0.75 \end{aligned}$$

# Independence

Events $A$ and $B$ are independent if $P(A, B) = P(A)P(B)$.

- Indepentent: $A$: first toss is HEAD; $B$: second toss is HEAD;

$$P(A, B) = 0.5 * 0.5 = P(A)P(B)$$

- Not Indepentent: $A$: first toss is HEAD; $B$: first toss is HEAD;

$$P(A, B) = 0.5 \neq P(A)P(B)$$

# Independence

Events $A$ and $B$ are conditionally independent given $C$ if

$$P(A, B|C) = P(B|C)P(A|C)$$

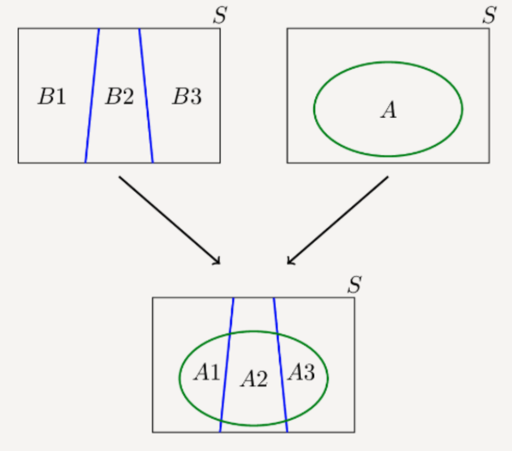
Consider two coins [1]: A regular coin and a coin which always outputs HEAD or always outputs TAIL.
$A$=The first toss is HEAD; $B$=The second toss is HEAD; $C$=The regular coin is used. $D$=The other coin is used.
Then $A$ and $B$ are conditionally independent given $C$, but $A$ and $B$ are NOT conditionally independent given $D$.

[1] `www.probabilitycourse.com/chapter1/1_4_4_conditional_independence.php`

# Marginalization and Law of Total Probability

Law of Total Probability [2]

$$P(X) = \sum_Y P(X, Y) = \sum_Y P(X|Y)P(Y)$$




[2] www.probabilitycourse.com/chapter1/1_4_2_total_probability.php

# Bayes' Rule

Bayes' Rule:

$$P(A|B) = \frac{P(B|A)P(A)}{P(B)}$$

$$P(\theta|x) = \frac{P(x|\theta)P(\theta)}{P(x)}$$

$$\text{Posterior} = \frac{\text{Likelihood} \times \text{Prior}}{\text{Evidence}}$$

$$\text{Posterior} \propto \text{Likelihood} \times \text{Prior}$$

# Bayes' Example

Suppose you have tested positive for a disease. What is the probability you actually have the disease?

This depends on the prior probability of the disease:

- $P(T = 1|D = 1) = 0.95$ (likelihood)
- $P(T = 1|D = 0) = 0.10$ (likelihood)
- $P(D = 1) = 0.1$ (prior)

So $P(D = 1|T = 1) =?$

# Bayes' Example

Suppose you have tested positive for a disease. What is the probability you actually have the disease?

$P(T=1|D=1)=0.95$ (true positive)

$P(T=1|D=0)=0.10$ (false positive)

$P(D=1)=0.1$ (prior)

So $P(D=1|T=1)=?$
Use Bayes' Rule:

$$P(D=1|T=1)=\frac{P(T=1|D=1)P(D=1)}{P(T=1)}=\frac{0.95\times 0.1}{P(T=1)}=0.51$$

$$P(T=1)=P(T=1|D=1)P(D=1)+P(T=1|D=0)P(D=0)$$
$$=0.95\times 0.1+0.1\times 0.90=0.185$$

# Random Variable

How do we connect sample spaces and events to data?

A random variable is a mapping which assigns a real number $X(\omega)$ to each observed outcome $\omega \in \Omega$

For example, let's flip a coin 10 times. $X(\omega)$ counts the number of Heads we observe in our sequence. If $\omega = HHTHTHHTHT$ then $X(\omega) = 6$.

# Discrete and Continuous Random Variables

Discrete Random Variables

- Takes countably many values, e.g., number of heads
- Distribution defined by probability mass function (PMF)
- Marginalization: $p(x) = \sum_y p(x, y)$

Continuous Random Variables

- Takes uncountably many values, e.g., time to complete task
- Distribution defined by probability density function (PDF)
- Marginalization: $p(x) = \int_y p(x, y) \mathrm{d}y$

# I.I.D.

Random variables are said to be independent and identically distributed (i.i.d.) if they are sampled from the same probability distribution and are mutually independent.

This is a common assumption for observations. For example, coin flips are assumed to be iid.

# Probability Distribution Statistics

Mean: First Moment, $\mu$

$$\mathbb{E}[X] = \sum_{i=1}^{\infty} x_i p(x_i) \qquad \text{(univariate discrete r.v.)}$$

$$\mathbb{E}[X] = \int_{-\infty}^{\infty} x p(x) dx \qquad \text{(univariate continuous r.v.)}$$

Variance: Second (central) Moment, $\sigma^2$

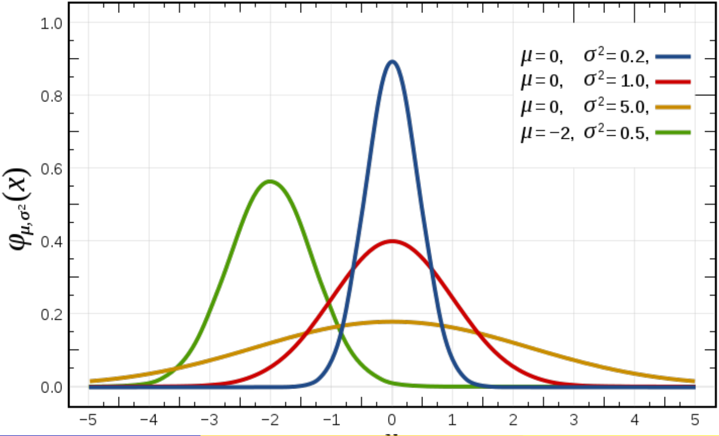
$$\begin{aligned} \mathrm{Var}[X] &= \int_{-\infty}^{\infty} (x-\mu)^2 p(x) \mathrm{d}x \\ &= \mathbb{E}\left[(X-\mu)^2\right] \\ &= \mathbb{E}\left[X^2\right] - \mathbb{E}[X]^2 \end{aligned}$$

It is common to use capital letters such as $X$ to denote a random variable drawn from a distribution $p(x)$. That is why we wrote $\mathbb{E}[X]$ instead of $\mathbb{E}[x]$, but the latter may also be used sometimes. We may go back and forth between these two.

# Univariate Gaussian Distribution

Also known as the Normal Distribution, $\mathcal{N}(\mu, \sigma^2)$

$$\mathcal{N}(x|\mu, \sigma^2) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right)$$



# Multivariate Gaussian Distribution

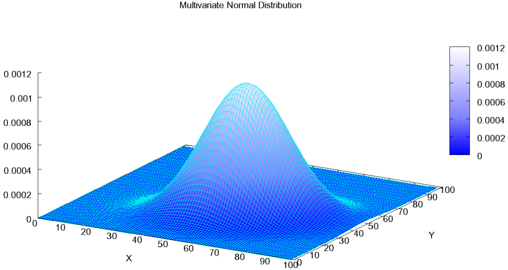
Multidimensional generalization of the Gaussian.

$\mathbf{x}$ is a D-dimensional vector

$\mu$ is a D-dimensional mean vector

$\Sigma$ is a $D \times D$ covariance matrix with determinant $|\Sigma|$

$$\mathcal{N}(\mathbf{x}|\mu, \Sigma) = \frac{1}{(2\pi)^{D/2}} \frac{1}{|\Sigma|^{1/2}} \exp\left(-\frac{1}{2}(\mathbf{x}-\mu)^T \Sigma^{-1} (\mathbf{x}-\mu)\right)$$



# Covariance Matrix

Recall that $\mathbf{x}$ and $\mu$ are D-dimensional vectors
Covariance matrix $\Sigma$ is a matrix whose $(i, j)$ entry is the covariance

$$\begin{aligned}\Sigma_{ij} &= \mathbf{Cov}\left(\mathbf{X}_i, \mathbf{X}_j\right) \\ &= \mathbb{E}\left[(\mathbf{X}_i - \mu_i)(\mathbf{X}_j - \mu_j)\right] \\ &= \mathbb{E}\left[\mathbf{X}_i \mathbf{X}_j\right] - \mu_i \mu_j.\end{aligned}$$

Notice that the diagonal entries are the variance of each elements.
The covariant matrix has the property that it is symmetric and positive-semidefinite (this is useful for whitening).

# Inferring Parameters

We have data $X$ and we assume it is sampled from some distribution.
How do we figure out the parameters that "best" fit that distribution?
Maximum Likelihood Estimation (MLE)

$$\hat{\theta}_{MLE} = \underset{\theta}{\operatorname{argmax}} P(X|\theta)$$

Maximum A posteriori Probability (MAP)

$$\hat{\theta}_{MAP} = \underset{\theta}{\operatorname{argmax}} P(\theta|X)$$

# MLE for Univariate Gaussian Distribution

We are trying to infer the parameters mean $\mu$ and variance $\sigma^2$ of a univariate Gaussian Distribution:

$$\mathcal{N}(x|\mu, \sigma^2) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp(-\frac{1}{2\sigma^2}(x-\mu)^2).$$

The likelihood that our observations $X_1, \ldots, X_N$ were generated by a univariate Gaussian with parameters $\mu$ and $\sigma^2$ is

$$\text{Likelihood} = p(X_1, \ldots, X_N|\mu, \sigma^2) = \prod_{i=1}^{N} \frac{1}{\sqrt{2\pi\sigma^2}} \exp(-\frac{1}{2\sigma^2}(X_i-\mu)^2).$$

# MLE for Univariate Gaussian Distribution

For MLE we want to maximize this likelihood, which is difficult because it is represented by a product of terms

$$\text{Likelihood} = p(X_1, \ldots, X_N | \mu, \sigma^2) = \prod_{i=1}^{N} \frac{1}{\sqrt{2\pi\sigma^2}} \exp(-\frac{1}{2\sigma^2}(X_i - \mu)^2)$$

So we take the log of the likelihood so the product becomes a sum

$$\begin{aligned} \text{Log Likelihood} &= \log p(X_1, \ldots, X_N | \mu, \sigma^2) \\ &= \sum_{i=1}^{N} \log \frac{1}{\sqrt{2\pi\sigma^2}} \exp(-\frac{1}{2\sigma^2}(X_i - \mu)^2). \end{aligned}$$

Since log is monotonically increasing, their maximizers are the same, i.e. $\operatorname{argmax} \theta L(\theta) = \operatorname{argmax} \theta \log L(\theta)$.

# MLE for Univariate Gaussian Distribution

The log Likelihood simplifies to

$$\mathcal{L}(\mu, \sigma) = \sum_{i=1}^{N} \log \left[ \frac{1}{\sqrt{2\pi\sigma^2}} \exp(-\frac{1}{2\sigma^2}(X_i - \mu)^2) \right]$$
$$= -\frac{1}{2} N \log(2\pi\sigma^2) - \sum_{i=1}^{N} \frac{(X_i - \mu)^2}{2\sigma^2}$$

Which we want to maximize. How?

# MLE for Univariate Gaussian Distribution

To maximize we take the derivatives, set equal to 0, and solve:

$$\mathcal{L}(\mu, \sigma) = -\frac{1}{2} N \log(2\pi\sigma^2) - \sum_{i=1}^{N} \frac{(x_i - \mu)^2}{2\sigma^2}$$

Derivative w.r.t. $\mu$, set equal to 0, and solve for $\hat{\mu}$

$$\frac{\partial \mathcal{L}(\mu, \sigma)}{\partial \mu} = 0 \implies \hat{\mu} = \frac{1}{N} \sum_{i=1}^{N} X_i.$$

Therefore the $\hat{\mu}$ that maximizes the likelihood is the average of the data points, which is called the sample average or empirical expectation too.
Derivative w.r.t. $\sigma^2$, set equal to 0, and solve for $\hat{\sigma}^2$

$$\frac{\partial \mathcal{L}(\mu, \sigma)}{\partial \sigma^2} = 0 \implies \hat{\sigma}^2 = \frac{1}{N} \sum_{i=1}^{N} (X_i - \hat{\mu})^2.$$